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Department of
Theoretical and Applied Mechanics
CORNELL UNIVERSITY
ITHACA, NEW YORK

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MATHEMATICAL THEORY OF LAMINAR COMBUSTION XI:

Stability

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J.D. Buckmaster & G.S.S. Ludford

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## Foreward

This report is Chapter XI of the twelve in a forthcoming research monograph on the mathematical theory of laminar combustion. Chapter I-IV originally appeared as Technical Reports Nos. 77, 80, 82 & 85; these were later extensively revised and then issued as Technical Summary Reports No's 1803, 1818, 1819 & 1888 of the Mathematics Research Center, University of Wisconsin-Madison. References to I-IV mean the MRC reports.

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The findings of this report are not to be construed
as an official Department of the Army position unless
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# Chapter XI

## Stability

### 1. Scope

Flame stability is a subject of enormous breadth, which could easily fill a monograph by itself albeit a less analytical one than ours; both the mechanisms and manifestations of instability assume many forms. It has been studied at least since the observations of Higgins (1777) on singing flames, a phenomenon involving interaction of the kind described in Chapter V between an acoustic field and an oscillating flame. Such interactions were laboratory curiosities for many years, but lately they have assumed technological importance in the development of rocket motors and large furnaces. This type of instability is distinguished by being well understood (Chu,     ), at least qualitatively. Other types of instability are in general poorly understood and, in some cases, even in doubt, in the sense that instability is suspected of playing a role but no certain evidence is yet available. The following examples, though by no means constituting a complete list, convey some idea of the richness of instability phenomena in combustion.

The stability of burner flames depends on an appropriate interaction between flame and surroundings, in particular a flux of heat from the flame to the burner rim. The role of this flux in anchoring the flame and preventing blow off has already been mentioned in Chapter IX.

Propagation limits are important questions for premixed flames, steady sustained combustion being possible only for a certain range of the fuel to oxidant ratio. If a mixture is too rich or too lean it will not burn, and instability may well play a role in the matter. A possible related

phenomenon is quenching by heat loss or non-uniformities (Chapter X), but whether this is a question of existence or stability (or some combination of the two), is not yet known.

Flow is turbulent in the right circumstances and indeed turbulent combustion is of paramount importance in many technological applications. How burning influences the transition from laminar to turbulent flow is not well understood at present. Instability may also play a role in another transition, that from deflagration to detonation (DDT). A flame propagating through a mixture will, under appropriate stimulus, accelerate and change from being an isobaric deflagration wave into a detonation wave, with shock-like structure.

Instability is invariably invoked to eliminate branches of multiple responses for being physically unrealizable. Several examples are provided by the burning of a condensate (Chapter IV), and others by fuel- and mono-propellant-drop combustion (Chapters VI and VII).

Finally, a striking manifestation of instability is provided by cellular flames, which are discussed in detail by Markstein (1964). Under certain conditions a nominally plane wave displays cells, i.e. an unsteady, quasi-periodic, two-dimensional, transverse pattern (Fig. 1). On average the cells have a characteristic dimension, but large cells grow and divide while small cells shrink and vanish. The corresponding phenomenon in highly curved burner flames leads to beautiful polyhedral tips (Fig. 2), which sometimes are quite steady and sometimes rotate but never exhibit the unsteadiness of plane cellular flames. Cells have also been observed on a spherical flame spreading out from an ignition source (Istratov & Librovich 1969).

All these examples apply to premixed flames and some to diffusion flames, which introduce their own peculiar questions associated with the separation of the reactants. However, we shall be concerned only with the stability of the freely propagating premixed flame, and cellular flames will play a central role in the development. Such a narrow focus is dictated by the state of the art, analytically speaking: certainly it is the only area that has been explored by activation-energy asymptotics to any siginificant extent, a big attraction being the simple closed form of the unperturbed steady state. Even then formidable difficulties remain; that they have ben overcome is due to the tenacity with which Sivashinsky, the principal architect of the theory we shall describe, has pursued the problem.

In our opinion, the material in the present chapter, answering as it does questions that have long frustrated the combustion scientist, is one of the triumphs of activation-energy asymptotics.

## 2. Slowly Varying Perturbations of the Plane Wave

The motion of a slowly varying flame is governed by the basic equation (VIII.32), where $\underset{\sim}{v}_1$ is to be determined from a coupled hydrodynamic analysis. There the flame appears as a discontinuity across which the jump conditions (VIII.1,2,3) hold and on either side of which Euler's equations (VIII.34) are valid.

If the gas is quiescent for ahead of the flame there is a solution

$$\underset{\sim}{v}_1 = 0, \quad M_n = 1, \quad \kappa = 0, \tag{1}$$

namely the steady plane flame. Small perturbations of this stationary state, to be denoted by primes, then satisfy

$$\dot{M}'_n - \nabla_{\perp} \cdot \mathbf{v}'_{p1} - \frac{1}{\rho_1} \nabla^2_{\perp} F' + \frac{2\beta}{\rho_1} M'_n = 0 \, , \tag{2}$$

where

$$M'_n = \rho_1(v'_{n1} - F'_{\tau}) \, , \quad \beta = T^2_* \rho_1/b \tag{3}$$

and we have used $V = 1/\rho_1$. Without loss of accuracy, the $\xi$, $\eta$, $\zeta$-frame may be fixed at an arbitrary point of the undisturbed plane wave. The velocity perturbation must be determined by the hydrodynamic analysis mentioned above, a step to which we shall return shortly.

Only for one-dimensional disturbance of the flame surface can the velocity perturbations be neglected, as will become clear in the sequel. Nevertheless, doing so in general leads to a correct description of one of the disturbance modes as $k \to 0$ or $|\beta| \to \infty$. (The latter corresponds to small heat release, when the unit of temperature becomes small and hence $T_*$ large, or to a Lewis number close to one.) Sivashinsky (1977a) then speaks of the diffusional-thermal effect, since these are the processes left. With

$$\mathbf{v}'_1 = 0 \quad \text{and hence} \quad M'_n = -\rho_1 F'_{\tau} \, , \tag{4}$$

equation (2) admits spatially periodic solutions

$$F' = \rho_1 A \exp (\alpha\tau + ik\eta) \tag{5}$$

provided

$$(k^2 - \rho_1^2 \alpha^2 - 2\beta\rho_1\alpha)A = 0 \quad \text{i.e.} \quad \rho_1\alpha = -\beta \pm \sqrt{\beta^2 + k^2} \, , \tag{6}$$

where no loss of generality results from considering disturbances independent of $\zeta$.

For one-dimensional disturbances $k = 0$ the two roots are

$$\rho_1 \alpha = 0 , \quad -2\beta , \tag{7}$$

and the flame is unstable if and only if $\beta$ is negative, i.e. the Lewis number is greater than 1. This result was uncovered in our earlier discussion of plane flames (Sec. III.5). On the other hand, for $k \neq 0$ there is a positive root whatever the value of $\mathcal{L}$ so that instability always occurs. The conclusion is found to be correct when velocity perturbations are taken into account, although the smaller (in absolute value) of the roots (6), on which it is based, is spurious, i.e. not associated with any of the actual modes. On the other hand, the larger root does correspond to an actual mode in the limit $|\beta| \to \infty$ or $k \to 0$, though only to leading order. Thus there is a diffusional-thermal mode but it is not necessarily unstable. We turn now to the hydrodynamic problem.

The jump conditions (VIII.1,2,3) show that the state behind the plane flame is

$$P_2 = P_1 - Y_1 , \quad u_2 = Y_1 , \quad v_2 = 0 \tag{8}$$

while the perturbations on the two sides satisfy

$$p_2' = p_1' - 2(\sigma-1)(u_1'-F_\tau') , \quad u_2' = \sigma u_1' - (\sigma-1)F_\tau' , \quad v_2' = v_1' - Y_1 F_\eta' . \tag{9}$$

Here $u$ and $v$ are the velocity components in the $\xi$- and $\eta$- direction and

$$\sigma = T_2/T_1 = 1 + Y_1/T_1 \tag{10}$$

When account is taken of the convective accelerations due to the velocities $T_1 (= 1/\rho_1)$ and $T_2$ in front of and behind the discontinuity, Euler's equations (VIII.34) are seen to have the solutions

$$[p',u',v'] = [-(\rho_1\alpha+|k|),|k|,ik]P_1\exp(\alpha\tau+|k|\xi+ik\eta) \quad \text{for} \quad \xi < 0 \ ,$$

$$[p',u',v'] = [\rho_2\alpha-|k|,|k|,-ik]P_2\exp(\alpha\tau-|k|\xi+ik\eta)$$

$$+[0,\sigma|k|,-i\rho_1\alpha\,\mathrm{sgn}k]S\exp(\alpha\tau-\rho_2\alpha\xi+ik\eta) \quad \text{for} \quad \xi > 0 \ . \tag{11}$$

The P-terms form potential fields for the velocities while the S-terms, which are only solenoidal, take account of (convected) vorticity behind the flame. These solutions are appropriate for a flame surface (5), and substituting into the jump conditions (9) yields

$$\sigma[\rho_1\alpha+(2\sigma-1)|k|]P_1 + (\rho_1\alpha-\sigma|k|)P_2 - 2\sigma(\sigma-1)\rho_1\alpha A = 0 \ ,$$

$$\sigma|k|P_1 - |k|P_2 - \sigma|k|S - (\sigma-1)\rho_1\alpha A = 0 \ ,$$

$$|k|P_1 + |k|P_2 + \rho_1\alpha S - (\sigma-1)|k|A = 0 \ . \tag{12}$$

If the flame displacement $A$ were specified, the equations would determine the hydrodynamic fields on the two sides of the surface. In fact $A$ is related to $P_1$ for slowly varying flames, as the basic equation (2) shows.

Landau (1944) and, independently, Darrieus (1946) supposed, in the absence of a theory, that the flame speed is unaffected by the perturbations; which, as we shall see, is true in the theory of slowly varying flames for $k \to 0$ or $|\beta| \to \infty$. The response of the flame is thereby eliminated and it is appropriate to speak of the hydrodynamic effect. Setting the flame-speed

perturbation $u_1' - F_\tau'$ equal to zero gives

$$|k|P_1 - \rho_1 \alpha A = 0 , \tag{13}$$

and then the homogeneous system for $P_1$, $P_2$, $S$ and $A$ has a non-trivial solution if and only if

$$\rho_1 \alpha = \sigma|k| \quad \text{or} \quad (\sigma+1)\rho_1^2\alpha^2 + 2\sigma|k|\rho_1\alpha - \sigma(\sigma-1)k^2 = 0$$

$$\text{i.e.} \quad \rho_1\alpha = \sigma(-1 \pm \sqrt{1+\sigma-1/\sigma})|k|/(\sigma+1) . \tag{14}$$

The first root gives $P_1 = A = 0$ so that the stability of the flame is not involved, although disturbances behind the flame are amplified. However, A is non-zero for the second root, which is always positive in view of $\sigma > 1$, and we conclude that the flame is unstable. Note that the result is quite independent of the theory of slowly varying flames (even though $\xi$, $\eta$, $\zeta$ were used): the hydrodynamic effect is always destabilizing though, of course, it may only be present when the variations are slow.

We now come to the stability of the plane wave to slowly varying disturbances, including both the diffusional-thermal and hydrodynamic effects. The basic equation (2) provides

$$(k^2 + \rho_1\alpha|k| + 2\beta|k|)P_1 + (k^2 - \rho_1^2\alpha^2 - 2\beta\rho_1\alpha)A = 0 \tag{15}$$

as fourth equation of the homogeneous system, in place of (13), so that the condition for a non-trivial solution becomes

$$a_o\rho_1^3\alpha^3 + a_1\rho_1^2\alpha^2 + a_2\rho_1\alpha + a_3 = 0 , \tag{16}$$

where

$$a_o = \sigma+1 \ , \quad a_1 = (\sigma+1)(|k|+2\beta) \ , \quad a_2 = \sigma|k|[-(3\sigma-1)|k|+4\beta] \ ,$$
$$a_3 = -\sigma k^2[(3\sigma-1)|k|+2(\sigma-1)\beta] \ , \tag{17}$$

if $\rho_1\alpha = \sigma|k|$ is again discarded.

From their behavior as $k \to 0$ or $|\beta| \to \infty$ it is clear that one of the three modes may be called diffusional-thermal and the other two hydrodynamic. In either case the cubic becomes

$$(\rho_1\alpha+2\beta)[(\sigma+1)\rho_1^2\alpha^2 + 2\sigma|k|\rho_1\alpha - \sigma(\sigma-1)k^2] = 0 \tag{18}$$

so that the larger of the roots (6) and the two relevant roots (14) are obtained. Correspondingly the basic equation (15) reduces to

$$(\rho_1^2\alpha^2 + 2\beta\rho_1\alpha)A = 0 \ . \tag{19}$$

That the condition (19) agrees with the diffusional-thermal condition (6) only in the limit $k \to 0$ or $|\beta| \to \infty$ itself shows that the smaller root (6) is spurious. It predicts $\rho_1\alpha = 0(k^2/2\beta)$ for $k$ small or $|\beta|$ large, whereas the actual remaining modes have $\rho_1\alpha = 0(k)$.

The cubic shows that steady corrugations $(\alpha = 0)$ with wavenumber

$$|k| = -2(\sigma-1)\beta/(3\sigma-1) \tag{20}$$

are possible when $\beta$ is negative $(\sigma > 1)$. The result was first obtained by Sivashinsky (1973) but is of historical interest only since for that value of $k$ (and indeed all values) we shall see that there is at least one root of the cubic with positive real part, implying instability.

A result of the type (16), for a more complicated model, was first obtained by Eckhaus (1961) from a thin-flame analysis which did not use formal activation-energy asymptotics. Instead he adopted a formula of Mallard and LeChatelier (see Emmons, 1958, *p.585* which leads to the perturbations of flame temperature and speed being proportional, a relation concistent with the asymptotic result (VIII.30) after linearization. The first complete deriva-tion was given, somewhat heuristically, by Sivashinsky (1977b) and, rigorously, by Buckmaster (1977)

Necessary conditions for every root of the cubic (16) to have a negative real part are $a_1$, $a_3 > 0$. Stability would therefore require

$$-|k| < 2\beta < -(\frac{3\sigma-1}{\sigma-1})|k|$$

which, for $\sigma > 1$ as here, is a contradiction. We are forced to conclude that plane flames are unstable.

The instability occurs whatever the wavenumber $k$, but it would be a mistake to infer that all steady, slowly varying flames are unstable. Such an extrapolation implies taking the short-wave limit $k \to \infty$ (so as to screen out boundedness and curvature) and hence violating the assumption of slow variations on which the conclusion of instability for unbounded plane flames was based. In any event the consequences of linear instability are not necessarily catastrophic; the result may be a nonlinear, unsteady structure superimposed on the steady state without obliterating its essential character.

Even the instability of unbounded plane flames to long-wave disturbances cannot be considered definitive, since there may be stabilizing influences that have been ignored so far. Two such influences are gravity and curvature, which we now examine.

## 3. Buoyancy and Curvature

The influence that gravity has on flames by virtue of the induced buoyancy forces is clear from a commonplace observation: an inverted candle does not behave in a desirable fashion. A less obvious manifestation is the dependence of the vertical propagation limit (i.e. the dilution beyond which the mixture is not inflammable) on whether the propagation is upwards or downwards. Instability should play a role in such limits since for upward propagation the light burnt gas underlies the heavy unburnt gas, so that Taylor-type instability can be anticipated. (The characterization is not entirely accurate since the flame is not a material surface.)

A flame of characteristic dimension 1 cm. and speed 45 cm./sec. has approximately equal gravity and dynamic heads. These quantities are typical, and faster or smaller flames are correspondingly less influenced by gravity. Moreover, since flame thickness is typically much less than a millimeter (Sec. II.4), gravity can only be important in the hydrodynamic field of a nominally plane flame. Support for that conclusion comes from the derivation of the basic equation (VIII.32) of slowly varying flames, which does not use momentum balance (the only governing equation influenced by gravity, see Sec. I.2).

The unit of acceleration is $c_p M^3/\rho_c \lambda$, which is typically $2 \times 10^5$ cm./sec.$^2$. The dimensionless gravity force is then $5 \times 10^{-3}$ and we may reasonably consider it $O(1/\theta)$. The momentum equation (VIII.34) now gains a term $-\rho(g,0)$ on its right-hand side, where $g$ is positive for downward propagation, so that terms

$$-\rho_1 g \xi \quad \text{and} \quad -\rho_2 g \xi \tag{21}$$

are added to the pressures in front of and behind the flame, respectively.

The perturbations $p_1'$ and $p_2'$ in equation (9) are now the sums of terms coming from equation (11), as before, and terms

$$-\rho_1 g F' = -\rho_1^2 g A \exp(\alpha\tau + ik\eta) \ , \quad -\rho_2 g F' = -\rho_1^2 \frac{gA}{\sigma} \exp(\alpha\tau + ik\eta) \ ,$$

so that the first of conditions (12) is replaced by

$$\sigma[\rho_1\alpha + (2\sigma-1)|k|]P_1 + (\rho_1\alpha - \sigma|k|)P_2 - (\sigma-1)(2\sigma\rho_1\alpha - \rho_1^2 g)A = 0 \ . \quad (22)$$

Clearly there is no limiting effect on the diffusional-thermal mode but there is on the hydrodynamic modes, the roots (14) becoming

$$\rho_1\alpha = \sigma[-1 \pm \sqrt{1 + (\sigma - 1/\sigma)(1 - \rho_1^2 g/\sigma|k|)}]|k|/(\sigma+1) \ . \quad (23)$$

Since $g$ is negative for upward propagation, we immediately see that gravity is destabilizing when the hot gas underlies the cold. When the reverse is true, disturbances for which

$$|k| < \rho_1^2 g/\sigma \quad (24)$$

are stable. For a downwardly propagating flame, modes of sufficiently long wavelength are therefore stabilized.

That these conclusions do indeed hold is seen from the cubic (16) with terms

$$(\sigma-1)\rho_1^2 g|k|(\rho_1\alpha + |k| + 2\beta)$$

added to account for gravity. In either of the limits $k \to 0$ or $\beta \to \infty$ the cubic becomes

$$(\rho_1\alpha+2\beta)\,[\,(\sigma+1)\rho_1^2\alpha^2 + 2\sigma|k|\rho_1\alpha - \sigma(\sigma-1)k^2 + (\sigma-1)\rho_1^2 g|k|\,]\ =\ 0 \qquad (25)$$

so that the diffusional-thermal mode is unaffected but the modifications (23) of the hydrodynamic modes are obtained. For the latter, the stabilizing effect is most clearly seen for long waves, when

$$\rho_1\alpha\ =\ \pm i\rho_1\sqrt{\frac{(\sigma-1)g}{\sigma+1}}\ |k|^{1/2} - \frac{\sigma}{\sigma-1}\ |k| + 0(k^{3/2})\quad. \qquad (26)$$

We conclude that for $\mathcal{L} > 1$ gravity cannot completely stabilize even long waves, since the relevant root (7) is a creature of purely one-dimensional disturbances and these are unaffected by the hydrodynamics; but for $\mathcal{L} < 1$ it can.

As $k \to \infty$ the cubic reduces to

$$(\rho_1\alpha+|k|)\,[\,(\sigma+1)\rho_1^2\alpha^2 - \sigma(3\sigma-1)k^2]\ =\ 0\ , \qquad (27)$$

which is independent of $g$: short waves are unaffected by gravity, as expected. Note how instability persists in the limit: there is always a positive root, though the other two are negative.

We now examine the effect of curvature, but only on the limit of the diffusional-thermal mode. A full treatment is quite complicated, much more so than for a plane flame, and in any event gravity stabilizes the other modes (at least for long wavelengths). Attention will be confined to a plane source or sink flow

$$u\ =\ c/r\ ,\quad v = 0 \qquad (28)$$

where $\theta r$ is radial distance and $u$, $v$ are polar components. The undisturbed flame is then circular, with curvature

$$\kappa = -1/\rho_1 c \tag{29}$$

determined by the requirement that the mass flux is 1 at it.

Such a flame corresponds to the solution

$$\underset{\sim}{v}_{p1} = 0 , \qquad M_n = 1 , \qquad V = 0 \tag{30}$$

of the basic equation (VIII.32), and perturbations of this steady state satisfy

$$\dot{M}'_n - \underset{\sim}{\nabla}_{\perp} \cdot \underset{\sim}{v}'_{p1} - \kappa V' + \frac{2\beta}{\rho_1} M'_n = 0 . \tag{31}$$

Here $v'_{p1}$ is due to the displacement of the flame in the source or sink flow, disturbances of which have been ignored. It is sufficient to consider plane disturbances, for which the flame may be written

$$r = \rho_1 |c| + F'(\phi, \tau) \tag{32}$$

in polar coordinates $r$, $\phi$. Then

$$V' = \mp F'_\tau , \quad M'_n = \mp (F'/\rho_1 c + \rho_1 F'_\tau) , \quad \underset{\sim}{\nabla}_{\perp} \cdot \underset{\sim}{v}'_{p1} = \pm F'_{\phi\phi}/\rho_1^3 c^2 , \tag{33}$$

according as $c$ is positive or negative, and the behavior of modes

$$F' = \rho_1 A \exp(\alpha \tau + in\phi) \tag{34}$$

is determined by

$$\rho_1^2 \alpha^2 + 2(\beta + 1/\rho_1 c)\rho_1 \alpha + (2\beta - n^2/\rho_1 c)/\rho_1 c = 0 \tag{35}$$

as

$$\rho_1 \alpha = -(\beta + 1/\rho_1 c) \pm \sqrt{(\beta + 1/\rho_1 c)^2 - (2\beta - n^2/\rho_1 c)/\rho_1 c} \quad . \tag{36}$$

Clearly the roots are always real.

To discuss the result we introduce the wavenumber

$$k = n/\rho_1 |c| \tag{37}$$

and note that the formula (6) is recovered in the limit $|c| \to \infty$ with $k$ fixed. Since the smaller root was then found to be spurious, we shall discard the corresponding root here. The larger root is negative if and only if $\beta > -1/\rho_1 c$, so that curvature is a stabilizing influence for the source flow but destabilizing for the sink flow. The conclusion must however be considered tentative until incorporation of the hydrodynamics shows that we are in fact dealing with the limit $\beta \to \infty$ or $k \to 0$.

The requirement that $c$ be positive corresponds to a flame that is concave towards the fresh mixture, as for a closed burner flame. Such flames were found for $\mathcal{L} < 1$ (Sec. IX.3) and our theory predicts stability of the diffusional-thermal mode. On the other hand, the open flames found for $\mathcal{L} > 1$ correspond to $c < 0$ and instability is inferred. Extrapolations of this kind are, however, little more than guesses.

## 4. Stability of Near-Equidiffusional Plane Flames.

The conclusion that modes of short wavelength will always be unstable according to the theory of slowly varying flames, while not a valid one, nevertheless stands in sharp contrast to experimental results. We therefore

turn to near-equidiffusional flames for clarification, and here compromise is necessary. On the scale of the flame thickness slowly varying flames are almost plane and almost steady; these characteristics allow the fluid mechanics, in particular variable density, to be incorporated without difficulty. By contrast, the continuity and momentum equations do not simplify when the flame is near-equidiffusional, so that perturbations of even the plane flame lead to non-constant coefficients. For that reason most discussion of near-equidiffusional stability has relied on the constant-density approximation, as embodied in equations (VIII.62, 63), and with the single exception of Sec. 6 that is the framework adopted here.

For flames in a quiescent gas, the governing equations reduce to

$$\partial(T,h)/\partial t = \nabla^2(T,h+\lambda T) ; \tag{38}$$

to these are added the jump conditions (VIII.64). The steady plane wave corresponds to the solution

$$\left. \begin{array}{l} T = T_1 + Y_1 e^n , \quad h = -\lambda Y_1 n e^n \quad \text{for} \quad n < 0 , \\[2mm] T = T_* = H_1 , \quad h = 0 \quad \text{for} \quad n > 0 \end{array} \right\} \tag{39}$$

where

$$n = x+t , \tag{40}$$

the flame sheet being located at $x = -t$ [cf. equations (IX.29,30)]. Its stability to small perturbations is our first concern.

The class of disturbances is limited: $O(I)$ perturbations of $T$ are admitted provided they are balanced by $O(1)$ perturbations of $Y$ which keep the sum of order $\theta^{-1}$. It follows that a finding of stability (in

contrast to instability) is not definitive. Nevertheless the conclusions are very convincing and it seems unlikely that an analysis dealing with a larger class of disturbances would alter the picture substantially.

Such perturbations satisfy

$$\partial (T',h')/\partial t = \nabla^2 (T', h'+\lambda T') \tag{41}$$

on both sides of the disturbed flame sheet

$$n = F'(y,t) \ . \tag{42}$$

The appropriate solution is

$$T' = 0 \quad \text{for} \quad n > 0 \tag{43}$$

behind the flame sheet. Continuity of T and h, plus the jump conditions on their derivatives, yield

$$\left.\begin{aligned}
T' &= -Y_1 F' \ , \quad [h'] = -\lambda Y_1 F' \ , \\
\partial T'/\partial n &= -Y_1 F' + Y_1 h'/2T_*^2 \ , \quad [\partial h'/\partial n] = \lambda Y_1 h'/2T_*^2 - 2\lambda Y_1 F'
\end{aligned}\right\} \text{for} \quad n = 0 \ , \tag{44}$$

where the right-hand sides of the last two equations are to be evaluated for n = 0+, i.e. on the hot side of the flame sheet.

Variations in the z-direction may be ignored without loss in generality of the conclusions. The solution corresponding to

$$F' = A \exp (\alpha t + iky) \tag{45}$$

is

$$(T',h') = (B,C+\lambda(\kappa_1^2-k^2)Bn/(1-2\kappa_1))\exp(\alpha t+iky+\kappa_1 n) \quad \text{for} \quad n < 0 ,$$

$$(46)$$

$$h' = D\exp(\alpha t+iky+\kappa_2 n) \quad \text{for} \quad n > 0 ,$$

where

$$\kappa_{1,2} = \frac{1}{2}[1 \pm \sqrt{1+4\alpha+4k^2}] , \quad -\pi < \arg(1+4\alpha+4k^2) \le \pi . \quad (47)$$

To be admissible the solution must vanish far ahead of the flame sheet and be bounded far behind, i.e. Re $\kappa_1$ must be positive and Re $\kappa_2$ non-positive. For Re $\alpha \ge 0$ (i.e. the unstable modes of interest and their neutral limits) these requirements are met automatically, so we need not concern ourselves with them further. Equations (44) now yield a homogeneous system for A, B, C, D which has a non-trivial solution only if

$$(1-\kappa_1)(1-2\kappa_1)^2+\tilde{\lambda}[(1-\kappa_1)^2-k^2] = 0 , \quad \tilde{\lambda} = Y_1\lambda/2T_*^2 , \quad (48)$$

a result first obtained by Sivashinsky (1977a).

This condition determines all those modes, i.e. values of $\alpha$, corresponding to any given $\tilde{\lambda}$ and k. For Re $\alpha \ge 0$ we can be sure the mode is admissible, the boundaries of regions of instability Re $\alpha > 0$ being curves Re $\alpha = 0$. There are just three such curves, namely

the $\tilde{\lambda}$-axis: $k = 0$ , $\qquad\qquad (49)$

the parabola P: $4k^2 = -(\tilde{\lambda}+1)$ , $\qquad\qquad (50)$

the branch B of the curve $(1+12k^2)\tilde{\lambda}^2 - 4(1+8k^2)\tilde{\lambda} -$
$- 8(1+8k^2)^2 = 0$ for which $\tilde{\lambda} \ge 16/3$ . $\qquad (51)$

For the first two $\mathrm{Im}\,\alpha = 0$ also.

The regions of instability can now be determined from explicit results for small $k$, namely

$$\alpha = -(\tilde{\lambda}+1)k^2 + \tilde{\lambda}^2(\tilde{\lambda}-3)k^4 + 0(k^6) \ , \quad [\tilde{\lambda}^2-4\tilde{\lambda}-8+\tilde{\lambda}\sqrt{\tilde{\lambda}(\tilde{\lambda}-8)}]/32 + 0(k^2)$$

$$(52)$$

where both square roots are implied for $0 < \tilde{\lambda} < 8$. Clearly there is a mode with $\mathrm{Re}\,\alpha > 0$ for

$$\tilde{\lambda} < -1 \quad \text{or} \quad > 2(1+\sqrt{3}) \tag{53}$$

respectively, but not otherwise, so that the stability regions are as marked in Fig. 3. Note how misleading results would be obtained by considering only one-dimensional disturbances: for $k = 0$ the first of the values (52) vanishes and no instability is predicted for $\tilde{\lambda} < -1$. The resulting stability condition $\tilde{\lambda} \leq 2(1+\sqrt{3})$ can be considered a refinement of $\mathcal{L} < 1$ for slowly varying disturbances [Sec. 2, cf. equation (7)].

For each value of $k$ there is a band of Lewis numbers, always including $-1 \leq \tilde{\lambda} \leq 16/3$ and hence $\mathcal{L} = 1$, for which the flame is stable. The conclusion is in accord with that for slowly varying disturbances, where instability is found for all $\mathcal{L} \neq 1$ (Sec. 2) and, moreover, provides a possible explanation of why stable flames are observed, a possibility that makes near-equidiffusional flames of great practical importance. To be sure, in the limit $\theta \rightarrow \infty$ the stability band in $\mathcal{L}$ is vanishingly small, but in practice $\theta$ is only moderately large. Since Lewis numbers are invariably close to one, it is conceivable that all flames adequately described by activation-energy asymptotics may be considered near equi-diffusional.

On the boundary  B  the neutral stability modes are oscillatory, i.e.
Im $\alpha \neq 0$. The boundary is relevant to mixtures whose deficient component
is heavy and therefore has relatively small diffusivity (e.g. for lean
hydrocarbon or rich hydrogen flames in air), but it has received no attention
to date.

On the parabolic boundary and on the $\bar{\lambda}$-axis the neutral modes are non-
oscillatory, i.e.  Im $\alpha = 0$.  (In fact, all modes inside the parabola are
non-oscillatory too.)  For any  $\tilde{\lambda} < -1$  and wavenumber in the range

$$0 < |k| < \sqrt{-(\tilde{\lambda}+1)/2} \qquad (54)$$

there is a mode that will grow.  Moreover, since  Re $\alpha$ vanishes at the ends
of the range there must be an intermediate value, namely

$$k_i = [\frac{4+15\tilde{\lambda}-9\tilde{\lambda}^2 + (4-6\tilde{\lambda})\sqrt{1-3\tilde{\lambda}}}{108\tilde{\lambda}}]^{1/2} \quad , \qquad (55)$$

for which the growth is a maximum.  Although the instability does not favor
one wavelength to the exclusion of all others, an arbitrary disturbance can
be expected to grow in such a way that the length  $1/k_i$  plays a significant
role.  As we noted in Sec. 1 it is characteristic of cellular flames, which
are usually observed when the deficient-component is light ($\lambda$ negative), for
many modes to contribute to the structure, but nevertheless an average dimension
can be assigned to the cells.

That the stationary mode corresponding to the parabola has relevance to
cellular flames is supported by the behavior of the flame-temperature perturba-
tion

$$2T_*^2(1-\kappa_1)\Lambda \exp iky = T_*^2(\sqrt{1+4k^2} - 1)\Lambda \exp i(ky+\pi)(k = \sqrt{-(\tilde{\lambda}+1)/2})$$

$$(56)$$

corresponding to the displacement

$$F' = A \exp iky \quad , \tag{57}$$

with which it is exactly out of phase. Viewed from the burnt mixture, the crests are colder than the troughs. We would therefore expect the crests to be the least luminous part, which is in fact a well-known characteristic of cellular flames.

## 5. Cellular Flames

The linear analysis of the previous section predicts that infinitesimal disturbances will grow without bound when $\tilde{\lambda}$ is less than $-1$, but there is strong evidence that the growth actually leads to cellular flames. In that case, nonlinearities must limit the growth; the nature of that interaction will now be investigated. Attention will be focussed on the neighborhood of $\tilde{\lambda} = -1$ by setting

$$\lambda = -\lambda_0 \mp \varepsilon \quad \text{with} \quad \lambda_0 = 2T_*^2/Y_1 \quad , \tag{58}$$

where $\varepsilon > 0$ is vanishingly small. Only the top sign on $\varepsilon$ is of interest here, corresponding to the linearly unstable waves in that neighborhood (i.e. the long ones), but the bottom sign will be used later. Such a focus can be treated by a perturbation analysis in $\varepsilon$. The stability boundary (50) shows that these waves have a wave-number $k = 0(\varepsilon^{1/2})$ at most, while the result (52) shows that they grow no faster than $\alpha = 0(\varepsilon^2)$. Accordingly the slow variables

$$\eta = \epsilon^{1/2} y , \quad \tau = \epsilon^2 t \tag{59}$$

are appropriate for their description.

For fixed $\epsilon$, an infinitesimal disturbance will grow according to the linear analysis until its size makes some nonlinear term comparable to the linear terms. Its amplitude is then $0(\epsilon)$ so that, following Sivashinsky (1977b), we take the flame sheet as

$$x = -t + \epsilon F(\eta, \tau) . \tag{60}$$

Nonlinearity has certainly intruded by then, as is seen from the resulting flame speed

$$W = 1 - \epsilon^3 (F_\tau + \overline{\phantom{-}} F_\eta^2 / 2) + 0(\epsilon^6) , \tag{61}$$

but it could conceivably intrude elsewhere for amplitude $o(\epsilon)$; the consistency *of the analysis stemming from the assumption* (60) shows that it does not. If now the coordinate

$$n = x + t - \epsilon F(\epsilon^{1/2} y, \epsilon^2 t) \tag{62}$$

is introduced in place of $x$, so that $n = 0$ is the flame sheet, then derivatives in equations (38) become

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial n} , \quad \frac{\partial}{\partial y} = -\epsilon^{3/2} F_\eta \frac{\partial}{\partial n} + \epsilon^{1/2} \frac{\partial}{\partial \eta} , \quad \frac{\partial}{\partial t} = (1 - \epsilon^3 F_\tau) \frac{\partial}{\partial n} + \epsilon^2 \frac{\partial}{\partial \tau}$$

while the normal derivative in the jump conditions is

$$(1 + \frac{1}{2} \epsilon^3 F_\eta^2) \frac{\partial}{\partial n} - \epsilon^2 F_\eta \frac{\partial}{\partial n} .$$

Perturbation expansions are now introduced for $T$, $h$ and $F$, for example

$$T = T_o + \varepsilon T_1 + \varepsilon^2 T_2 + \varepsilon^3 T_3 + \ldots \,, \tag{63}$$

where the coefficients are functions of $n$, $\eta$, $\tau$. A sequence of problems then results, whose solutions are subject to the requirements that $T_1 = T_2 = T_3 = \ldots = 0$ for $n > 0$, that conditions as $n \to -\infty$ are undisturbed and that exponential growth as $n \to +\infty$ is disallowed (though algebraic growth cannot be prevented). At the fourth problem, for $T_3$ and $h_3$, the nonlinear equation (75) for $F_o$ is obtained as a solvability condition.

The first problem is

$$\mathcal{L}(T_o) = \mathcal{L}(h_o) - \lambda_o \partial^2 T_o / \partial n^2 = 0 \quad \text{for all } n \,, \tag{64}$$

$$[T_o] = [h_o] = 0 \,, \quad \lambda_o [\partial T_o / \partial n] = [\partial h_o / \partial n] = -\lambda_o Y_1 \exp(h_o / 2 T_*^2) \quad \text{at } n = 0 \,,$$

where

$$\mathcal{L} \equiv \partial^2 / \partial n^2 - \partial / \partial n \tag{65}$$

As expected, its solution is the steady plane wave

$$T_o = \begin{cases} T_1 + Y_1 e^n & , \\ T_* = H_1 & , \end{cases} \qquad h_o = \begin{cases} \lambda_o Y_1 n e^n \\ 0 \end{cases} \quad \text{for } n \lessgtr 0 \,. \tag{66}$$

no confusion will result from the double use of $T_1$, since we shall show immediately that the expansion coefficient is identically zero. In fact the second problem is easily seen to be

$$\mathcal{L}(T_1) = 0 \;, \quad \mathcal{L}(h_1) - \lambda_o \partial^2 T_1/\partial n^2 = \begin{cases} \pm Y_1 e^n & \text{for } n \lessgtr 0 \;, \\ 0 & \end{cases}$$

$$[T_1] = [h_1] = 0 \;, \quad [\partial T_1/\partial n] = -h_1/\lambda_o \;, \quad [\partial h_1/\partial n] = -(h_1 \pm Y_1) \quad \text{at } n = 0 \tag{67}$$

when the independence of $T_o$ and $h_o$ from $n, \tau$ is used; so that

$$T_1 = \begin{cases} 0 \;, \\ 0 \;, \end{cases} \qquad h_1 = \begin{cases} \pm Y_1 n e^n \\ 0 \end{cases} \quad \text{for } n \lessgtr 0 \tag{68}$$

is obtained without using the jump condition on $\partial h_1/\partial n$, which is then automatically satisfied. The information obtained so far enables the third problem to be written

$$\mathcal{L}(T_2) = \begin{cases} Y_1 F_{o\eta\eta} e^n \;, \\ 0 \;, \end{cases} \quad \mathcal{L}(h_2) - \lambda_o \partial^2 T_2/\partial n^2 = \begin{cases} \lambda_o Y_1 F_{o\eta\eta} n e^n & \text{for } n \lessgtr 0 \;, \\ 0 & \end{cases}$$

$$[T_2] = [h_2] = 0 \;, \quad [\partial T_2/\partial n] = -h_2/\lambda_o \;, \quad [\partial h_2/\partial n] = -h_2 \quad \text{at } n = 0 \;. \tag{69}$$

Again, the solution

$$T_2 = \begin{cases} Y_1 F_{o\eta\eta} n e^n \;, \\ 0 \;, \end{cases} \qquad h_2 = \begin{cases} \lambda_o Y_1 F_{o\eta\eta} (1+n^2) e^n \\ \lambda_o Y_1 F_{o\eta\eta} \end{cases} \quad \text{for } n \lessgtr 0 \tag{70}$$

is determined without using the jump condition on $\partial h_2/\partial n$, which is then satisfied automatically.

Finally we reach the fourth problem, which can now be written

$$\mathcal{L}(T_3) = \begin{cases} Y_1(a_1 n + a_2) e^n \\ 0 \end{cases} \;, \quad \mathcal{L}(h_3) - \lambda_o \partial^2 T_3/\partial n^2 = \begin{cases} Y_1(a_3 n^2 + a_4 n + a_5) e^n & \text{for } n \lessgtr 0 \;. \\ \lambda_o Y_1 a_1 \end{cases}$$

$$[T_3] = [h_3] = 0 \;, \quad [\partial T_3/\partial n] = Y_1 F_{on}^2/2 - h_3/\lambda_o \;, \quad [\partial h_3/\partial n] = -h_3 \mp Y_1 F_{o\eta\eta} + \lambda_o Y_1 F_{on}^2$$

$$\text{at } n = 0 \;, \tag{71}$$

$$a_1 = -F_{o\eta\eta\eta} \quad , \quad a_2 = -F_{o\tau} + F_{1\eta\eta} - F_{o\eta}^2 \quad , \quad a_3 = \lambda_o a_1 \quad ,$$

$$a_4 = -\lambda_o (F_{o\tau} - F_{o\eta\eta\eta} - F_{1\eta\eta} + F_{o\eta}^2) \pm 2F_{o\eta\eta} \quad , \quad a_5 = -\lambda_o (F_{o\tau} + F_{o\eta\eta\eta} + F_{o\eta}^2) \pm 2F_{o\eta\eta} \quad \Bigg\} \tag{72}$$

The solution is determined without the last jump condition as

$$T_3 = \begin{cases} Y_1 (b_1 n^2 + b_2 n) e^n \quad , \\ 0 \quad , \end{cases} \qquad h_3 = \begin{cases} Y_1 (b_3 n^3 + b_4 n^2 + b_5 n + b_6) e^n \\ Y_1 (b_7 n + b_6) \end{cases} \qquad \text{for } n \lessgtr 0 \, , \tag{73}$$

where

$$b_1 = a_1/2 \quad , \quad b_2 = a_2 - a_1 \quad , \quad b_3 = \lambda_o a_1/2 \quad ,$$

$$b_4 = (a_4 + \lambda_o a_2 - 2\lambda_o a_1)/2 \quad , \quad b_5 = a_5 - a_4 + \lambda_o (a_2 + a_1) \quad , \qquad \Bigg\} \tag{74}$$

$$b_6 = -\lambda_o (F_{o\tau} - F_{o\eta\eta\eta} - F_{1\eta\eta} + F_{o\eta}^2/2) \quad , \quad b_7 = -\lambda_o a_1 \quad .$$

Note that $h_3$ must be allowed to increase algebraically. The jump condition on $\partial h_3/\partial n$ is not satisfied automatically but requires

$$F_{o\tau} + 4F_{o\eta\eta\eta} \pm F_{o\eta\eta}/\lambda_o + F_{o\eta}^2/2 = 0 \, . \tag{75}$$

This result is due to Sivashinsky (1977b), who generalized it to

$$F_{o\tau} + 4\nabla^4 F_o \pm \nabla^2 F/\lambda_o + (\nabla F_o)^2/2 = 0 \tag{76}$$

for variations in both the y- and z- directions.

The equation has an interesting implication for the flame speed. While the expression (61) is purely kinematical, its approximation

$$W = 1 + \varepsilon^3 (4F_{onnnn} \pm F_{onn}/\lambda_o) \tag{77}$$

is a profound consequence of the diffusional-thermal processes. Markstein (1964), in an attempt to modify Landau's conclusion that all flames are hydro-dynamically unstable, assumed that the flame-speed perturbation is proportional to curvature and so incorporated the term in $F_{onn}$ but not the other.

The linearized form of equation (75) has solutions proportional to $\exp(\alpha\tau/\varepsilon^2 + ik\eta/\varepsilon^{1/2})$ if

$$\alpha = \varepsilon k^2/\lambda_o - 4k^4 , \tag{78}$$

which coincides with the first of approximations (52) in the neighborhood of $\lambda = -\lambda_o$, $k = 0$ . This result is graphed in Fig. 4. The maximum growth rate occurs for

$$k_i = \sqrt{\varepsilon/8\lambda_o} , \tag{79}$$

which is an approximation to the general formula (55). The connection of this maximum with the characteristic cell size has already been noted. Existing discussion of the nonlinear equation (75) is limited to numerical computations (Sivashinsky & Michelson, 1977). Integration of the initial-value problem with periodic boundary conditions (using a period large compared to $k_i^{-1}$) leads to results like those in Fig. 5 when the plus sign is taken in the equation. (For the minus sign there is decay to zero, corresponding to linear stability.) The nonlinearity prevents unlimited growth of the disturbance and the resulting structure, being quasiperiodic with characteristic dimension $k_i^{-1}$, is highly suggestive of cellular flames.

## 6. Hydrodynamic Effects

The analysis of Sec. 5, leading as it does to a balance of small effects, provides a framework within which several generalizations can be discussed so long as the additional effects are appropriately small. Here we shall consider weak interaction of the flame with the hydrodynamic field. A rigorous derivation of the governing equation replacing (75) requires analysis similar to that in Sec. 5. A more transparent derivation can be made by plausible arguments.

The constant-density model is studied because it is simple and yet retains some of the key physical ingredients that govern flame behavior. Nevertheless it is a rational asymptotic limit $(\sigma \to 1)$, albeit one of limited practical interest since density changes across flames are rarely small. When $\sigma$ is close to 1, the hydrodynamic effects can be incorporated in a rational fashion as perturbations just strong enough to influence the flame behavior described in Sec. 5. Since we are only concerned with wavelengths $0(\varepsilon^{-1/2})$, the field outside the flame sheet has a dual structure similar to that of slowly varying flames (Sec. VIII.3): hydrodynamic disturbances described on the scale $n = 0(\varepsilon^{-1/2})$ must be matched with those in the diffusion zone, where $n = 0(1)$.

The flame speed (61) is now augmented by

$$(u_1' - \varepsilon^{3/2} F_n v_1')[1 + 0(\varepsilon^3)]$$

where $u_1'$, $v_1'$ are the components of the velocity disturbance immediately ahead of the flame sheet. If this is to be a perturbation just comparable to the existing one, then $u_1'$ must be $0(\varepsilon^3)$; in which case $v_1'$ (being of the same order) makes a smaller contribution and $W$ is, to order $\varepsilon^3$, simply augmented by $u_1'$.

The problem now is to express $u_1'$ in terms of $F_0$, which involves treating the flame sheet in the same way as the hydrodynamic discontinuity was in Sec. 2. Indeed, the weakness of the velocity perturbation ensures that Euler's equations are satisfied even in the diffusion zone, so that we may read off the result

$$P_1 = (\sigma-1)A/2 \qquad (80)$$

of solving the jump conditions (12) when $k$ and $\alpha$ are appropriately small and $\sigma$ is close to 1. Equation (80) relates $u_1'$ and $F_0$ when they are proportional to $\exp(\alpha t + iky)$ and hence the Fourier transforms of general $u_1'$ and $F_0$ (since $\alpha$ is not involved). It follows that

$$u_1'(\eta,\tau) = \frac{(\sigma-1)\varepsilon}{4\pi\rho_1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |k| e^{ik(y-\tilde{y})} F_0(\tilde{\eta},\tau) d\tilde{y} \, dk$$

$$\qquad (81)$$

$$= \frac{(\sigma-1)\varepsilon^{3/2}}{2\pi\rho_1} \int_{-\infty}^{\infty} \frac{F_{0\eta}(\tilde{\eta},\tau)}{\eta-\tilde{\eta}} d\tilde{\eta},$$

where

$$\sigma = 1 + O(\varepsilon^{3/2}) \qquad (82)$$

if $u_1'$ is to be $O(\varepsilon^3)$.

So far we have been concerned with modifying the kinematical result (61) for the flame speed; now we come to the result (77) of the diffusional-thermal processes, which is found to be unchanged. Certainly it is plausible that such weak density changes do not affect these processes. That being the case, the generalization

$$F_{o\tau} + 4F_{o\eta\eta\eta\eta} \pm F_{o\eta\eta}/\lambda_o + F_{o\eta}^2/2 + \gamma \int_{-\infty}^{\infty} \frac{F_{o\eta}(\tilde{\eta},\tau)}{\tilde{\eta}-\eta}\, d\tilde{\eta} = 0 \qquad (83)$$

of equation (75) follows, where

$$\gamma = (\sigma-1)/2\pi\rho_1\epsilon^{3/2} . \qquad (84)$$

The destabilizing effect of disturbances in the hydrodynamic field can be seen from considering small $F_o$. Under linearization, i.e. on dropping $F_{o\eta}^2/2$, there are solutions $\exp(\alpha\tau+ik\eta)$ with

$$\alpha = -4k^4 \pm k^2/\lambda_o + \pi\gamma|k| , \qquad (85)$$

and Fig. 6 shows graphs of these curves. As expected from the last section, for $\gamma = 0$ the lower sign gives stable modes only, but the upper sign involves unstable modes. For $\gamma > 0$, i.e. when the hydrodynamic field is disturbed, both signs give unstable modes. The possibility of stabilizing the modes again by gravity can also be seen. Gravity changes the velocity by an amount proportional to the displacement of the flame sheet (cf. Sec. 3), which corresponds to adding a constant to the righ-hand side of equation (85). If the constant is negative (i.e. gravity points from burnt to unburnt gas) and sufficiently large, then the curves in Fig. 6 are translated downwards into the quadrant $\alpha < 0$.

Sivashinsky & Michelson (1977) have treated the full equation (83) also, in particular for the plus sign when the hydrodynamics is an extra destabiliz-ing effect. Results similar to Fig. 5 are obtained, except that now the large wrinkles themselves contain several small wrinkles. We shall consider just two limits of the equation, both corresponding to significantly larger

departures of $\lambda$ from $-\lambda_o$ than of $\sigma$ from 1, i.e.

$$\delta = (\sigma-1)\epsilon^{-3/2} \to 0 \quad . \tag{86}$$

If changes remain on the scale of $\eta$ and $\tau$ then the hydrodynamic term drops out and equation (75) is recovered in the limit. This tendency was found by Michelson & Sivashinsky (1977b) in their numerical work.

If, however, changes occur on the slower scales $\delta\eta$ and $\delta^2\tau$ then the fourth derivative drops out and the equation reduces to

$$F_{o\tau} \pm F_{o\eta\eta}/\lambda_o + F_{o\eta}^2/2 + \gamma \int_{-\infty}^{\infty} \frac{F_{o\eta}(\tilde{\eta},\tau)}{\tilde{\eta}-\eta} d\tilde{\eta} = 0 \quad . \tag{87}$$

Interest now centers on the minus sign, i.e. the destabilizing effect of the hydrodynamics when there is stability otherwise. Michelson & Sivashinsky's computations show that from initial data of long period (but otherwise arbitrary) there emerges at large times a steady progressive wave of the same period. For such a wave $F_{o\tau}$ is replaced by a constant $- V$ and, except near points of relatively large curvature, $F_{o\eta\eta}$ may be neglected. The shape is then described by the nonlinear integral equation

$$F_{o\eta}^2/2 + \gamma \int_{-\infty}^{\infty} (\frac{1}{\tilde{\eta}-\eta} - \frac{1}{\tilde{\eta}}) F_{o\eta}(\tilde{\eta}) d\tilde{\eta} = 0 \tag{88}$$

for the slope $F_{o\eta}$. Here $\eta$ is measured from a point where the slope vanishes, so that

$$V = \gamma \int_{-\infty}^{\infty} \frac{F_{o\eta}(\cdot)}{\tilde{\eta}} d\tilde{\eta} \quad . \tag{89}$$

The equation expresses constancy of the flame speed, as Landau assumed in his stability analysis.

A continuous periodic solution for $F_o$, obtained numerically, is shown *(the value of V/y is 1.18)* in Fig. 7; (A solution for any other period can be obtained from it by *without changing V* scaling $\eta$) The discontinuities in $F_{o\eta}$ at $\pm\pi$ are smoothed out by the neglected $F_{o\eta\eta}$-term, whose inclusion perturbs the flame speed. We are therefore led to the suggestion that, under more general circumstances, the destabilizing effect of hydrodynamics first discovered by Landau does not result in chaotic structure but just a wrinkling of the nominal flame configuration. Such fine wrinkles are sometimes observed on actual flames (Markstein, 1964).

## 7. Curved Cellular Flames.

The cellular pattern in Fig. 1 is formed by a flame located in a large (10 cm. diameter) vertical tube and, consistent with the results of Sec. 5, is highly unsteady. By contrast, cellular flames stabilized on burners are usually steady, which suggests a classical bifurcation phenomenon similar to Taylor cells in cylindrical Couette flow. At a critical value of some parameter disturbances of a single finite wavenumber become unstable, and beyond that value there is an alternative (stable) steady state. Here we shall establish curvature as such a parameter, leading to a steady cellular pattern.

As in Sec. 3, attention will be confined to the plane flow

$$u = c/r , \quad v = 0 \tag{90}$$

with $c$ now positive only (source flow); $r$ is measured on the scale of the flame thickness. If disturbance of the hydrodynamic field is again neglected then equations (38) govern provided the replacement

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \frac{c}{r}\frac{\partial}{\partial r}$$

·is made. These possess a steady solution satisfying the jump conditions (VIII.64) in which the flame sheet is located at

$$r = c \tag{91}$$

and

$$T = T_1 + Y_1(r/c)^c \ , \qquad h = \lambda Y_1(r^c/c^{c-1})\ln(c/r) \quad \text{for} \quad r < c \ ,$$

$$T = H_1 \ , \quad h = 0 \quad \text{for} \quad r > c \ . \tag{92}$$

Such is the undisturbed flame.

The result (91) implies that the dimensionless burning rate is 1, i.e. the dimensional burning rate is the same as that of an adiabatic plane flame. There is an apparent contradiction with previous sections, where we saw a change in burning rate with curvature, which is resolved on recognizing change in curvature as the determining factor. The essential process is transverse diffusion, which is absent from the cylindrical flame because of symmetry, and its presence is detected by changes in curvature. The simplified treatment to be presented next will ensure that only disturbances in curvature influence the burning velocity.

The objective is to admit curvature disturbances just strong enough to modify equation (75). Since the $F_{\eta\eta}$-term corresponds to curvature $O(\epsilon^2)$, we are therefore led to introduce

$$R = \epsilon^2 c \tag{93}$$

as the $O(1)$ scaled radius of the undisturbed flame sheet. An $O(\varepsilon^{-1/2})$ change in $y$ nearby they corresponds to a change $O(\varepsilon^{3/2})$ in the polar angle $\phi$, so that we also introduce

$$\omega = \varepsilon^{-3/2}\phi \tag{94}$$

and take the disturbed position of the flame sheet to be

$$r = \varepsilon^{-2}R + \varepsilon f(\omega,\tau) \tag{95}$$

in a polar representation. The corresponding displacement from the tangent to the nominally circular flame sheet is then

$$F_o = \frac{1}{2} R\omega^2 + f(\omega,\tau) \tag{96}$$

in the notation of earlier sections, where

$$\omega = \eta/R \tag{97}$$

to sufficient accuracy.

The requirement that the existing curvature should not influence the flame velocity suggests modifying the result (77) to read

$$\begin{aligned}
W &= 1 + \varepsilon^3[4F_{o\eta\eta\eta\eta} \pm (F_{o\eta\eta} - R^{-1})/\lambda_o] \\
&= 1 + \varepsilon^3[4R^{-4}f_{\omega\omega\omega\omega} \pm R^{-2}f_{\omega\omega}/\lambda_o] \; .
\end{aligned} \tag{98}$$

To see that the modification agrees with Sec. X.3 we note that for a flat flame it reads

$$W = 1 \mp \varepsilon^3 R^{-1}/\lambda_o = 1 + (\overline{\lambda + \lambda_o}/\lambda_o)\partial v/\partial y \tag{99}$$

where v is here the velocity in the y-direction. However, the radial
flow is locally just a weak simple strain superimposed on a uniform flow,
so that the result (99) is recovered by letting the strain rate become small
in Sec. X.3. It is no accident then that $\lambda = -\lambda_0$ plays a role both in
the effect of straining flows on a flame and in linear stability analysis.

A modification of the kinematical result (61) arises because the flame
sheet moves in a non-uniform velocity field (without disturbing the field).
The accompanying changes $(-\epsilon^3 F_0/R, \epsilon^{3/2}\eta/R)$ in fluid velocity at the sheet
lead to the new approximation

$$W = 1 - \epsilon^3 [F_{0\tau} + F_{0\eta}^2/2 + R^{-1}(F_0 - \eta F_{0\eta})]$$

$$= 1 - \epsilon^3 (f_\tau + f_\omega^2/2R^2 + f/R) \quad . \tag{100}$$

Comparing the results (98) and (100) establishes

$$f_\tau + 4R^{-4} f_{\omega\omega\omega\omega} \pm R^{-2} f_{\omega\omega}/\lambda_0 + R^{-1}f + R^{-2}f_\omega^2/2 = 0 \tag{101}$$

as the fundamental equation, governing the radial perturbation  f  in the
position of the flame sheet. The term $R^{-1}f$ arises from the perturbation
in radial velocity at the flame, so that for nominally spherical flames
(where the velocity is proportional to $r^{-2}$) the term is replaced by $2R^{-1}f$
(Sivashinsky, 1978). We shall pursue the stability of the cylindrical
flame; the spherical flame requires only minor changes, including generaliza-
tion of the ω-derivatives (which is also needed here for variations in the
z-direction).

For linear stability we set  f  proportional to  $\exp(\alpha\tau + i\nu\omega)$  and neglect
the quadratic term to obtain

$$\alpha = \pm R^{-2} \nu^2 / \lambda_o - 4R^{-4} \nu^4 - R^{-1} \tag{102}$$

where, although it is strictly a discrete parameter (being $\varepsilon^{3/2}$ times an integer), $\nu$ effectively takes all real values. (If variations in the z-direction are admitted, with wave number $k$, then $\nu^2$ is replaced by $\nu^2 + R^2 k^2$ and the conclusions are modified in an obvious way.) As $R \to \infty$ *with*

$$R^{-1} \nu = k \tag{103}$$

held fixed the result (78) is recovered; and finite $R$ clearly has a stabilizing effect. For $\lambda > -\lambda_o$ curvature only argments the existing stability, but for $\lambda < -\lambda_o$ it can overcome the instability at sufficiently small wavenumbers $k$ (see Fig. 8). For all $R$ the maximum growth rate is attained for the value

$$k_c^2 = 1/8\lambda_o \ , \tag{104}$$

cf. equation (79), and it is zero when $R$ has the value

$$R_c = 16\lambda_o^2 \ . \tag{105}$$

for $R \leq R_c$ there is stability, but for $R > R_c$ there is a band of unstable waves.

The stability regions in the plane of $\sqrt{\lambda_o} |k|$ and $R/\lambda_o^2$ are shown in Fig. 9. As $R$ increases through $R_c$ a small band of unstable wavenumbers appears, suggesting a classical bifurcation. The corresponding Landau equation (cf. Matkowsky, 1970) is obtained in the usual way be setting

$$\varepsilon^2 = (R - R_c)/R_c \ , \quad f = \varepsilon(f_o + \varepsilon f_1 + \varepsilon^2 f_2 + \dots) \ , \quad \tau = \tau^*/\varepsilon^2 \ , \tag{106}$$

with

$$f = \epsilon A \cos \nu_c \omega \quad \text{for} \quad \tau^* = 0 . \tag{107}$$

To leading order we find that

$$f_o = A_o(\tau^*) \cos(\nu_c \omega) , \quad A_o(0) = A \tag{108}$$

and then that secular terms appear in $f_2$ unless

$$A_o' = A_o(9 - \lambda_o^2 A_o^2)/144\lambda_o^2 . \tag{109}$$

It follows that the amplitude asymptotes to the constant value

$$A_o = \pm 3/\lambda_o \quad \text{according as} \quad A \lessgtr 0 . \tag{110}$$

We conclude that the cylindrical flame sheet is stable for $R \leq R_c$ but unstable for $R > R_c$; and that for small positive values of $R - R_c$ there is a stable cellular configuration, with amplitide proportional to $(R-R_c)^{1/2}$, to which the originally cylindrical flame sheet tends after disturbance.

A characteristic feature of stationary cellular flames (including polyhedral flames) is that the crests, i.e. the portions of the flame that are convex towards the burnt gas, are sharper than the troughs. Here the variations in curvature are given by

$$f_{\omega\omega} = \epsilon\{-32\lambda_o^3[A_o\cos(\nu_c\omega)+\epsilon A_1\cos(\nu_c\omega)+\epsilon B_1\sin(\nu_c\omega)]-(64\lambda_o^4/9)\epsilon A_o^2\cos(2\nu_c\omega)\} + O(\epsilon^3)$$

$$\tag{111}$$

where $A_o$ has its ultimate value (110) as do the integration functions $A_1(\tau^*)$ and $B_1(\tau^*)$ in $f_1$. For $A_o > 0$ the crests lie at $\omega = 2n\pi/\nu_c$

and the troughs at $\omega = (2n+1)\pi/\nu_c$, where $n$ is an integer. Clearly the $0(\varepsilon^2)$ harmonic increases the magnitude of the curvature at the crests and decreases it at the troughs, consistent with observations.

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Fig. D.1 in Markstein (1964).

Fig.1. Cellular flames.

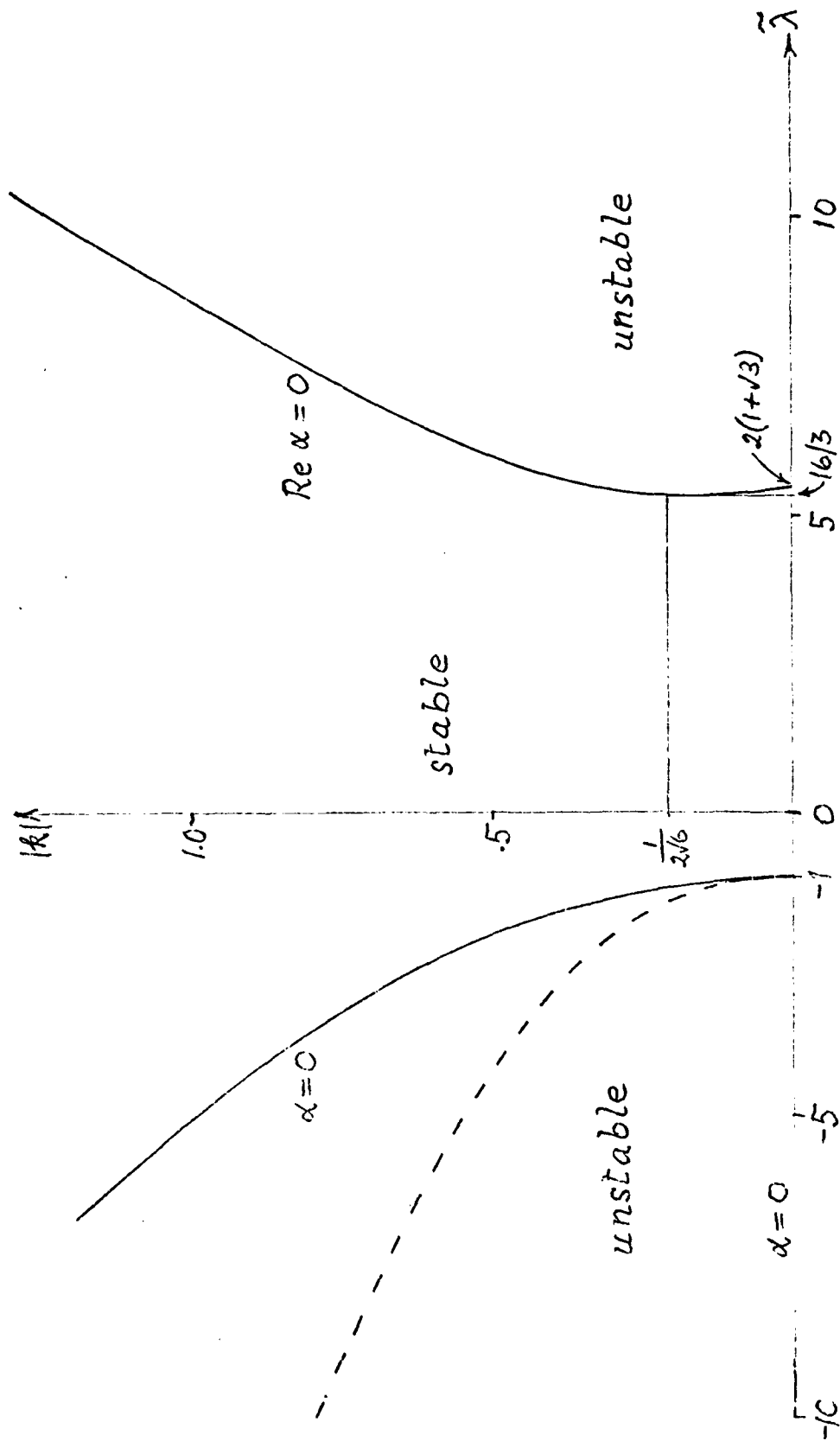Fig. 149 in Lewis & von Elbe (1961).

Fig. 2. Polyhedral flame tips.

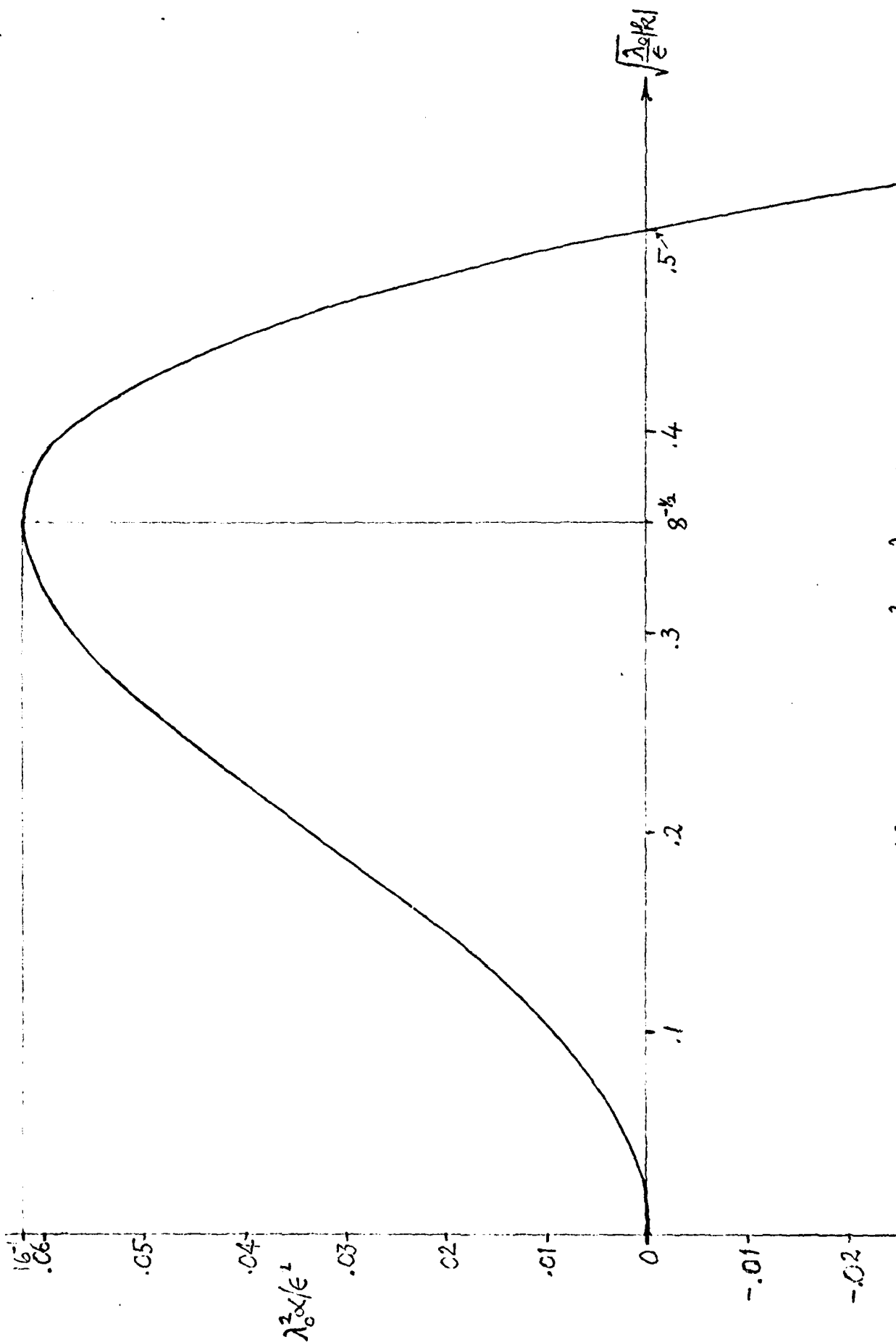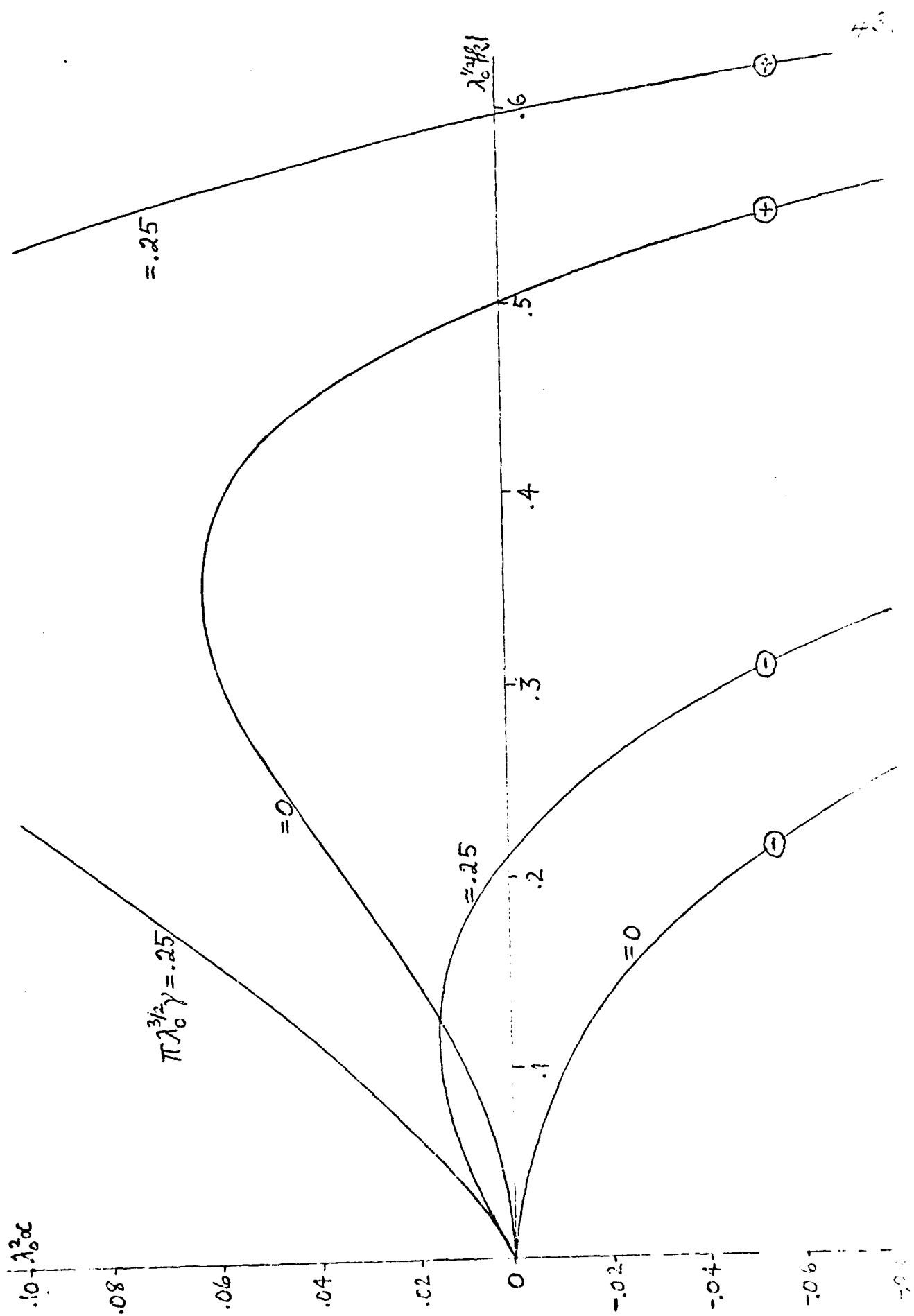Fig. 3. _Stability regions for near equi-diffusion. Maximum growth on curve_ − − − − −.

Fig.4.  *Growth rate near* $\lambda = -\lambda_0$.

Fig. 1 in Michelson & Sivashinsky (1977)

Fig.5 Evolution of cellular structure near λ=λ: integration of equation (75) for plus sign.

Fig. 6. $\alpha$ versus $k$ according to equation (85); signs correspond.
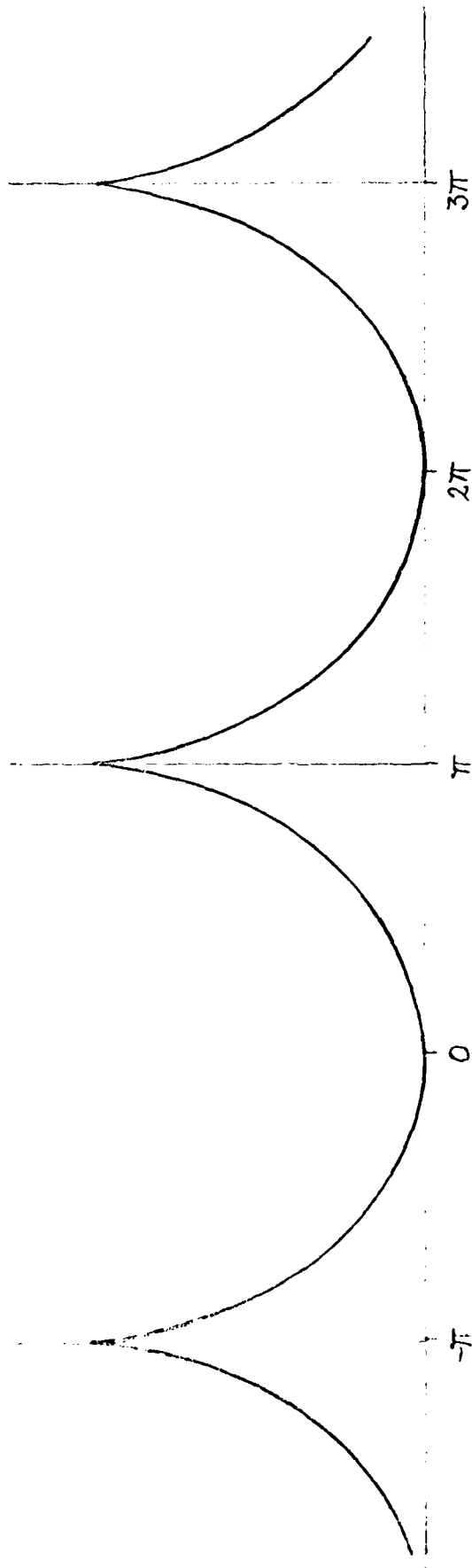
Fig. 7.   Wrinkled flame for $\lambda = -\lambda_0 + \epsilon$.

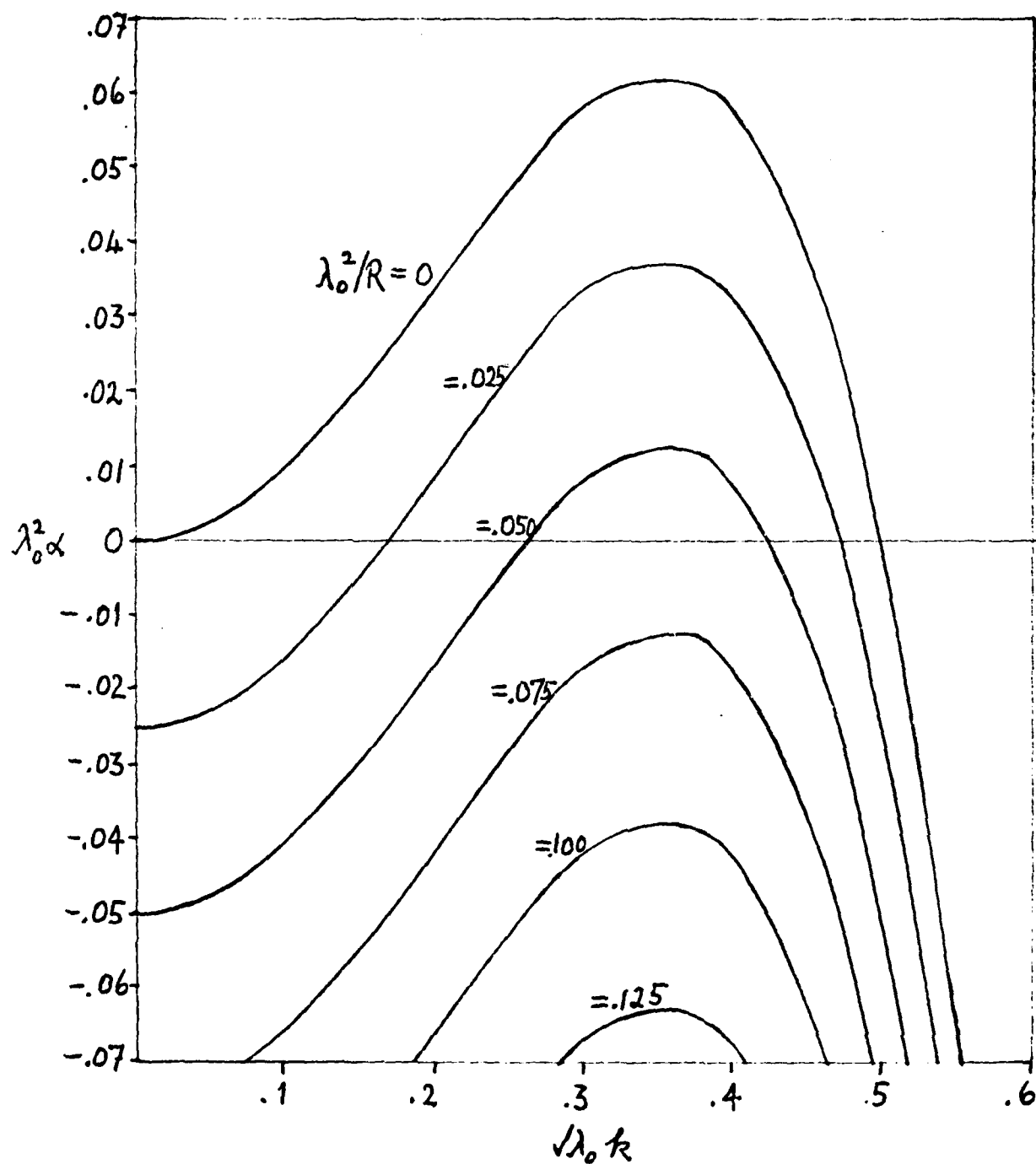Fig. 8. _Effect of curvature on the growth rate of disturbances near $\lambda = -\lambda_o$._
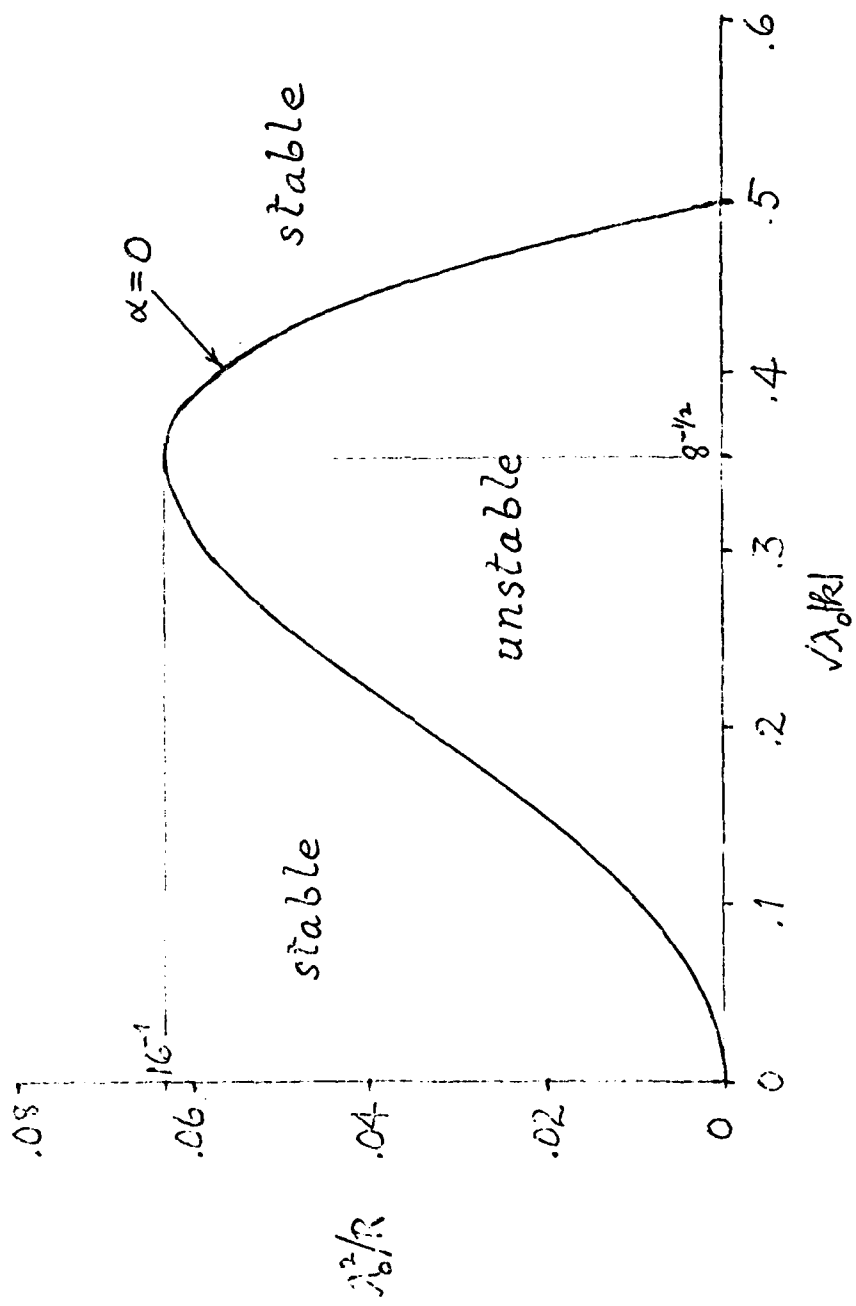
Fig. 9. Stability regions near $\lambda = -\lambda_o$.

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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

This report is Chapter XI of the twelve in a forthcoming research
monograph on the mathematical theory of laminar combustion. The investigation
is narrowed to the stability of freely propagating premixed flames, the only
area that has yielded to mathematical analysis so far. Special emphasis
is placed on the cellular flames that are found experimentally to be the result
of instability.

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